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Resolution of T. Ward's Question and the Israel–Finch Conjecture: Precise Analysis of an Integer Sequence Arising in Dynamics

JEFFREY GAITHER[1*], GUY LOUCHARD[2], STEPHAN WAGNER[3†] and MARK DANIEL WARD[4‡]

[1]Department of Mathematics, Purdue University, 150 North University Street, West Lafayette, IN 47907–2067, USA
(e-mail: gaither.16@mbi.osu.edu)
[2]Département d'Informatique, Université Libre de Bruxelles, CP 212, Boulevard du Triomphe, B-1050, Bruxelles, Belgium
(e-mail: louchard@ulb.ac.be)
[3]Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa
(e-mail: swagner@sun.ac.za)
[4]Department of Statistics, Purdue University, 150 North University Street, West Lafayette, IN 47907–2067, USA
(e-mail: mdw@purdue.edu)

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Dedicated to the memory of Philippe Flajolet

We analyse the first-order asymptotic growth of

$$a_n = \int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi j x)\,dx.$$

The integer a_n appears as the main term in a weighted average of the number of orbits in a particular quasihyperbolic automorphism of a $2n$-torus, which has applications to ergodic and analytic number theory. The combinatorial structure of a_n is also of interest, as the 'signed' number of ways in which 0 can be represented as the sum of $\epsilon_j j$ for $-n \leqslant j \leqslant n$ (with $j \neq 0$), with $\epsilon_j \in \{0, 1\}$. Our result answers a question of Thomas Ward (no relation to the fourth author) and confirms a conjecture of Robert Israel and Steven Finch.

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* Current address: Mathematical Biosciences Institute, The Ohio State University, Jennings Hall 381, 1735 Neil Ave., Columbus, OH 43210
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1. Introduction and motivation

We analyse the precise first-order asymptotics of integer sequence **A133871** from the *On-Line Encyclopedia of Integer Sequences* [12]. The sequence (a_n) is defined by the definite integral

$$a_n := \int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi jx)\,dx, \quad n = 1, 2, 3, \ldots. \tag{1.1}$$

The first few values of a_n are

$$(a_n)_{n\geqslant 1} = (2, 4, 6, 10, 12, 20, 24, 34, 44, 64, \ldots).$$

Thomas Ward (no relation to the fourth author) introduced this sequence on OEIS in 2008, stating that [12]:

This quantity arises in some examples associated to the dynamical Mertens' theorem for quasi-hyperbolic toral automorphisms. The function being integrated to compute a_n vanishes on the set of points in the Farey sequence of level n. I am particularly interested in knowing how large the sequence is asymptotically.

In this paper, we answer Thomas Ward's question about the asymptotics of a_n. Our main result is an exact first-order asymptotic analysis of a_n. We heartily thank Thomas Ward for explaining to us the motivation of the problem [20]. We explain this motivation further in this section. (The origins of this problem can be traced back to at least 1969, as explained below.)

Israel conjectured that

$$\frac{\log(a_n 2^{-n})}{n} \to -0.3\ldots,$$

or equivalently, $\log a_n \sim (\log(2) - 0.3\ldots)n$ (see [6]). Finch has written a short manuscript [4] in which he gives the equivalent conjecture, without proof (see the top of Finch's page 3), that $a_n^{1/n} \sim 1.48\ldots \sim 2e^{-0.29\ldots}$. Our main result rigorously verifies the Israel–Finch conjecture and sharpens it, to give an exact first-order asymptotic characterization. Moreover, instead of a numerical approximation, we provide the exact values of the constants in the asymptotics.

In the OEIS entry, Steven Finch also points out that 'a_n = coefficient of $x^{n(n+1)/2}$ in the polynomial $(-1)^n \prod_{k=1}^{n}(1-x^k)^2$, and is the maximal such coefficient as well' [12]. Finch's observation about this equivalent representation can be seen, for instance, as follows. Since $4\sin^2(\pi jx) = (1 - e^{2\pi ijx})(1 - e^{-2\pi ijx})$, then the integrand of a_n is

$$\prod_{j=1}^{n}(1 - e^{2\pi ijx})(1 - e^{-2\pi ijx}).$$

Note that $\int_0^1 e^{2\pi i\ell x}\,dx = 0$ for all non-zero integers ℓ, and $\int_0^1 e^{2\pi i0x}\,dx = 1$. So it follows that, if we compute a_n by expanding $\prod_{j=1}^{n}(1 - e^{2\pi ijx})(1 - e^{-2\pi ijx})$ and then integrating, only the terms *in which the js sum to* 0 will contribute. Moreover, if the integral of a term is non-zero, it is either 1 or -1, depending on whether an even or odd number of js were involved in the product.

This observation leads to the combinatorial representation

$$a_n = [x^0] \prod_{j=1}^{n} (1 - x^j)(1 - x^{-j}),$$

where $[x^0]$ denotes the constant coefficient (see Finch [4]). We can therefore think of a_n combinatorially as the *signed* total number of $(2n)$-tuples $(\epsilon_{-n}, \ldots, \epsilon_{-1}, \epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^{2n}$ such that

$$\sum_{\substack{-n \leqslant j \leqslant n \\ j \neq 0}} \epsilon_j j = 0.$$

By 'signed' we are referring to a weighting scheme in which a $(2n)$-tuple

$$(\epsilon_{-n}, \ldots, \epsilon_{-1}, \epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^{2n}$$

provides a contribution of -1 to the *signed* total number if an odd number of ϵ_j are non-zero (*i.e.*, equal to 1), or provides a contribution of $+1$ to the *signed* total number otherwise.

Example. When $n = 4$, exactly 18 of the 8-tuples have an even number of non-zero terms (*i.e.*, of ones),

(0,0,0,0,0,0,0,0)	(1,1,1,1,1,1,1,1)		
(0,0,0,1,1,0,0,0)	(0,0,1,0,0,1,0,0)	(0,1,0,0,0,0,1,0)	(1,0,0,0,0,0,0,1)
(0,0,1,1,1,1,0,0)	(0,1,0,1,1,0,1,0)	(0,1,1,0,0,1,1,0)	(1,0,0,1,1,0,0,1)
(1,0,1,0,0,1,0,1)	(1,1,0,0,0,0,1,1)	(0,1,1,0,1,0,0,1)	(1,0,0,1,0,1,1,0)
(0,1,1,1,1,1,1,0)	(1,0,1,1,1,1,0,1)	(1,1,0,1,1,0,1,1)	(1,1,1,0,0,1,1,1)

and exactly 8 of the 8-tuples have an odd number of non-zero terms (*i.e.*, of ones),

(0,0,1,1,0,0,1,0)	(0,1,0,0,1,1,0,0)	(0,1,0,1,0,0,0,1)	(1,0,0,0,1,0,1,0)
(0,1,1,1,0,1,0,1)	(1,0,1,0,1,1,1,0)	(1,0,1,1,0,0,1,1)	(1,1,0,0,1,1,0,1).

Thus $a_4 = 18(+1) + 8(-1) = 10$.

The number of ways in which 0 can be written in an unweighted fashion as $\sum_{j=-n}^{n} \epsilon_j j$ is a well-understood quantity (see Clark [2], Entringer [3], Louchard and Prodinger [9], van Lint [18], and others). To the best of our knowledge, our derivation is the first result in which the contributions are *signed* according to the parity of the number of non-zero terms.

Paul Hanna also has a note in the OEIS entry [12] that a_n equals the sum of the squares of the coefficients in the polynomial $\prod_{k=1}^{n}(1 - x^k)$.

From a topological/dynamical perspective, the quantity a_n is the asymptotic coefficient in the weighted sum of the orbital numbers of a certain toral automorphism. The study of this automorphism is motivated by the search for a topological analogue to Mertens' prime number theorem. There is a strong structural similarity between the distribution

of the prime integers and that of the orbits of an automorphism acting on an n-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$. Specifically, the classical Mertens' theorem gives

$$\prod_{p \leqslant N} \left(\frac{1}{1 - 1/p} \right) \sim \log(N),$$

and we have the analogous hyperbolic toral diffeomorphism result

$$M_T(N) := \sum_{|\tau| \leqslant N} \frac{1}{e^{h|\tau|}} \sim \log(N), \tag{1.2}$$

where each $\tau = \{x, T(x), \ldots, T^k(x) = x\}$ denotes a closed orbit of length k (here k and $|\tau|$ are both used interchangeably to represent the length), and h represents the topological entropy. Noorani [11] showed that when T is merely an ergodic toral automorphism (such an automorphism is said to *quasihyperbolic*), we have

$$M_T(N) = m \log(N) + C_1 + o(1) \tag{1.3}$$

for some positive integer m, where a_n is precisely the m in (1.3) for a specific automorphism, which we will introduce in the next paragraph. Jaidee, Stevens and Ward [7] improved Noorani's estimate and also showed that the constant m is given by

$$m = \int_X \prod_{j=1}^{n} 4 \sin^2(\pi x_j)\, d_{x_1} \cdots d_{x_j}, \tag{1.4}$$

where X is found as follows. We first find all eigenvalues of modulus 1 of the matrix A defining the automorphism T, and then, if these eigenvalues are $e^{\pm 2\pi i\theta_1}, \ldots, e^{\pm 2\pi i\theta_t}$, we let $X \subset \mathbb{T}^d$ denote the closure of the set $\{(k\theta_1, \ldots, k\theta_t) : k \in \mathbb{Z}\}$ in $\mathbb{T}^d$.

The particular toral automorphism that gives rise to our a_n has been of interest since at least 1969; see Walters' analysis [19]. Walters introduced the automorphism given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

on the 4-torus $\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1$ for an example of an affine mapping between compact connected metric abelian groups, for which the mapping commutes only with continuous maps that are also affine. For this matrix A, the eigenvalues are

$$2 + \sqrt{3} \pm \sqrt{6 + 4\sqrt{3}} \quad \text{and} \quad 2 - \sqrt{3} \pm i\sqrt{4\sqrt{3} - 6}.$$

The latter two eigenvalues, $2 - \sqrt{3} \pm i\sqrt{4\sqrt{3} - 6}$, each have modulus 1. Thus, the θ_1 from the previous paragraph is

$$\theta_1 = \frac{1}{2\pi} \arctan\left(\frac{2 - \sqrt{3}}{\sqrt{-6 + 4\sqrt{3}}} \right).$$

Noorani [11] further cited A as an example of a strictly quasi-hyperbolic automorphism on the torus. Jaidee, Stevens and Ward [7] later considered A as their example of a toral

automorphism whose asymptotic coefficient $m = 6$ exceeded $2^t = 2$ (where $t = 1$ since A has $2 = 2t$ eigenvalues on the unit circle), and in the same work used A to define the automorphism $A^1 \oplus A^2 \oplus \cdots \oplus A^n$ on the $4n$-torus $\mathbb{T}^{4n}$. This is the defining context of our a_n: it is precisely the coefficient m of $\log(N)$ in (1.3), in the asymptotic growth of $M_T(n)$, when M_T is defined as in (1.2) with $\mathbb{T}^{4n}$ the torus under consideration and $A^1 \oplus \cdots \oplus A^n$ the quasihyperbolic automorphism. This choice of automorphisms and tori gives a particularly nice form of (1.4), in that all the x_j are the same variable and the region of integration X is just $[0, 1]$.

2. Main result

Our goal is to determine the asymptotic growth of the quantity

$$a_n = \int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi j x)\, dx.$$

Our main result is a precise first-order characterization of the asymptotic growth of a_n.

Theorem 2.1. *Let a_n be defined as above, and let $G(x)$ be the function defined as*

$$G(x) := \int_0^1 \log(\sin(\pi x t))\, dt.$$

Then there is a unique point $x_0 = 0.7912265710\ldots$ on $(0, 1)$ at which G attains its maximum $\max_{x\in(0,1)} G(x) = -0.4945295654\ldots$ on the unit interval $(0, 1)$. Furthermore, if r and C denote the constants

$$r := e^{2G(x_0)} = 0.3719264606\ldots, \quad \text{and} \quad C := \frac{4\sin(\pi x_0)}{x_0}\sqrt{\frac{\pi}{-G''(x_0)}} = 2.405745839\ldots,$$

then the first-order asymptotic growth of a_n is $C(4r)^n n^{-1/2}$, that is,

$$\lim_{n\to\infty} \frac{a_n}{C(4r)^n n^{-1/2}} = 1.$$

We also prove a pair of twin theorems, stated below. Theorem 2.1 follows directly from these two theorems.

Theorem 2.2. *Let C, r be defined as in Theorem 2.1. Then*

$$\lim_{n\to\infty} \frac{\displaystyle\int_{\left[0,\frac{1}{n}\right]\cup\left[\frac{n-1}{n},1\right]} \prod_{j=1}^{n} \sin^2(\pi j x)\, dx}{Cr^n n^{-1/2}} = 1.$$

Theorem 2.3. *Let r be defined as in Theorem 2.1. Then*

$$\int_{\frac{1}{n}}^{\frac{n-1}{n}} \prod_{j=1}^{n} \sin^2(\pi j x)\, dx = O(\rho^n),$$

for some $\rho < r$.

Recall from equation (1.1) that a_n is defined as

$$a_n := \int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi jx)\,dx, \quad n = 1, 2, 3, \ldots.$$

Theorem 2.2 precisely characterizes the dominant contribution to the integral that defines a_n; this dominant contribution comes from integrating over the region $[0, 1/n] \cup [(n-1)/n, 1]$. Theorem 2.3 says that the integral over the middle integral $[1/n, (n-1)/n]$ is (comparatively) negligible. The proofs of Theorems 2.2 and 2.3 will occupy the remainder of the paper. The fact that the majority of the contribution to the integral comes from $x \in [0, 1/n] \cup [(n-1)/n, 1]$ is illustrated in Figure 1.

3. Derivation of asymptotics: Proof of Theorem 2.2

We first note that because $\sin^2(\pi jx)$ is symmetric about $x = 1/2$, then by a change of variables, we can express

$$\int_{\left[0,\frac{1}{n}\right]\cup\left[\frac{n-1}{n},1\right]} \prod_{j=1}^{n} \sin^2(\pi jx)\,dx$$

as

$$\frac{2}{n}\int_0^1 \prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{n}x\right) dx.$$

We can write this new integrand

$$\prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{n}x\right)$$

as an exponential function, which will allow us to use Laplace's method.

Lemma 3.1. *Let*

$$G(x) := \int_0^1 \log(\sin(\pi xt))\,dt,$$

and let

$$h_n(x) := \frac{\prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{n}x\right)}{2ne^{2nG(x)}} \quad \textit{for } 0 < x \leqslant 1.$$

We can (continuously) extend the domain of $h_n(x)$ to $[0, 1]$ by defining $h_n(0) := \lim_{x\to 0^+} h_n(x)$. Then we have

$$h_n(x) = \frac{\sin(\pi x)}{x} + O\left(\frac{1}{n}\right),$$

and this holds uniformly for $x \in [0, 1]$.

We postpone the proof of Lemma 3.1 for the moment, in order to show the connection to Theorem 2.2. We first state Laplace's method.

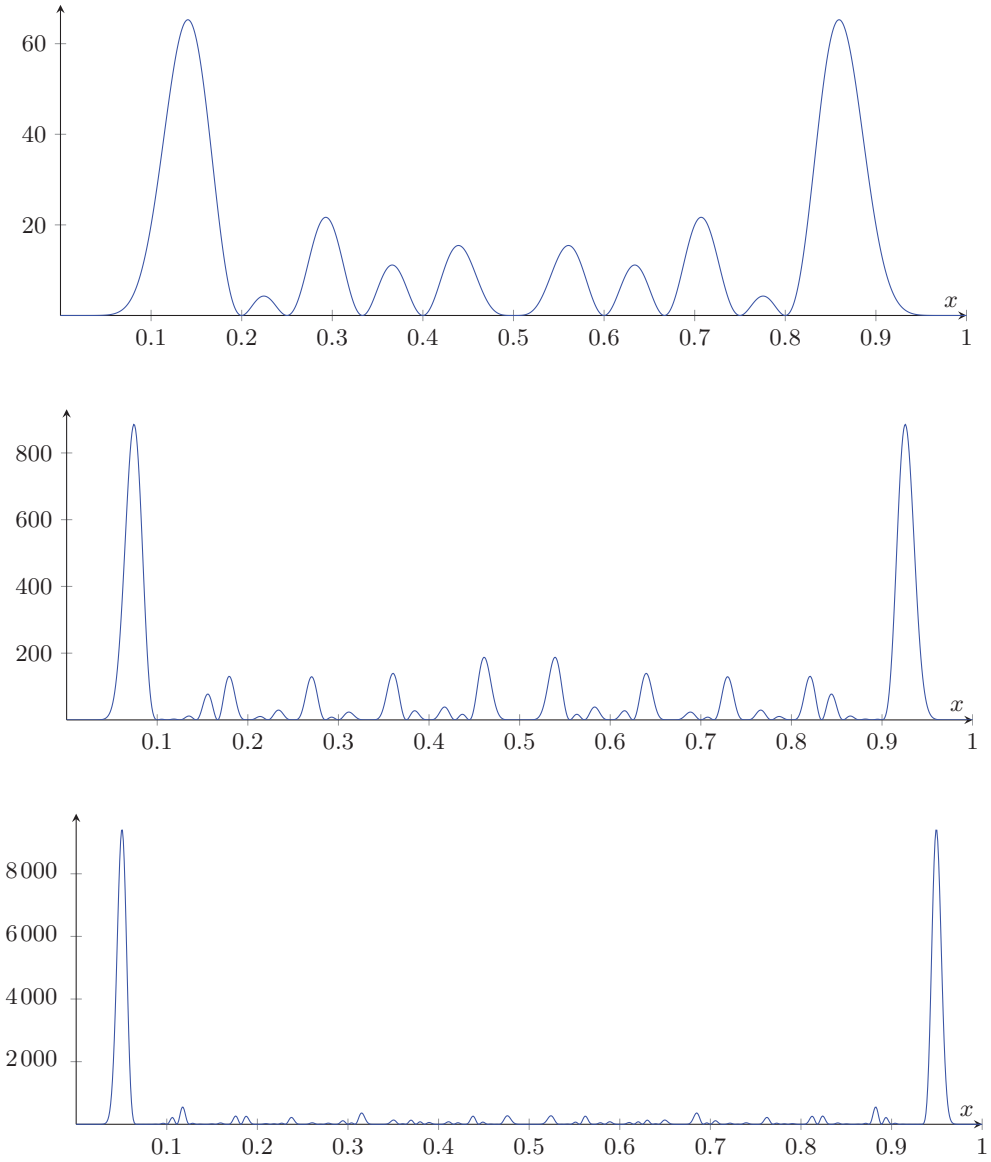

Figure 1. (Colour online) Plot of the integrand $\prod_{j=1}^{n} 4\sin^2(\pi jx)$ of $a_n := \int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi jx)\,dx$ for $n = 5, 10, 15$.

Theorem 3.2. (*Laplace's method: an easy extension of the usual version, e.g., as given in* [1] *or in Appendix B.6 of* [5].) *Let* $g : [a,b] \to \mathbb{R}$ *be twice-differentiable and concave on* $[a,b]$, *and suppose there is a unique point* $x_0 \in [a,b]$ *such that* $g(x_0) = \max_{x\in[a,b]} g(x)$. *Let* (f_n) *be a sequence of functions on* $[a,b]$ *which are uniformly bounded over all n, continuous on some common interval* $[x_0 - \epsilon, x_0 + \epsilon] \subset [a,b]$, *and uniformly convergent on* $[x_0 - \epsilon, x_0 + \epsilon]$ *to some* $f : [x_0 - \epsilon, x_0 + \epsilon] \to \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \frac{\int_a^b f_n(x)e^{ng(x)}\,dx}{f(x_0)e^{ng(x_0)}\sqrt{\frac{2\pi}{n(-g''(x_0))}}} = 1.$$

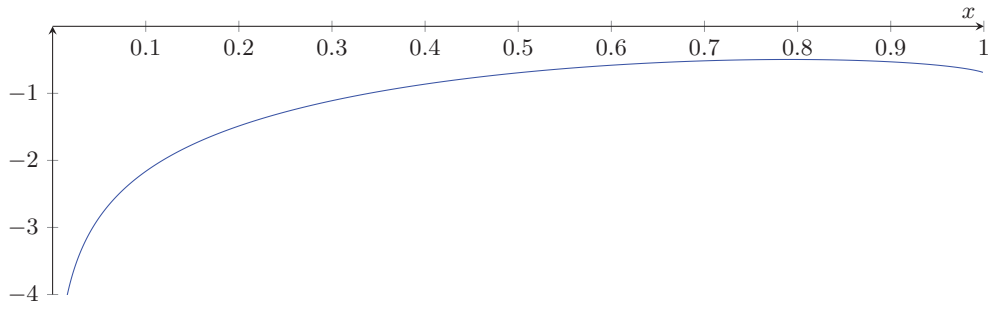

Figure 2. (Colour online) Plot of $G(x) := \int_0^1 \log(\sin(\pi x t))\,dt$ for $x \in (0,1)$.

We note that $G(x)$ is concave on $(0,1)$ since

$$G''(x) = -\int_0^1 \frac{\pi^2 t^2}{\sin^2(\pi x t)}\,dt < 0,$$

and it has the unique critical point $x_0 = 0.7912265710\ldots$, so that $G(x_0) > G(x)$ for all $x \in (0,1)$ with $x \neq x_0$. The function $G(x)$ is depicted in Figure 2.

Now we apply Laplace's method with

$$f_n := h_n, \quad f(x_0) := \frac{\sin(\pi x_0)}{x_0}, \quad [a,b] := [0,1], \quad g := 2G.$$

We obtain

$$\lim_{n\to\infty} \frac{\int_0^1 h_n(x) e^{2nG(x)}\,dx}{\frac{\sin(\pi x_0)}{x_0} e^{2nG(x_0)} \sqrt{\frac{\pi}{n(-G''(x_0))}}} = 1. \tag{3.1}$$

By definition, we have $r := e^{2G(x_0)}$ and

$$C := \frac{4\sin(\pi x_0)}{x_0} \sqrt{\frac{\pi}{-G''(x_0)}}.$$

Also

$$\int_{\left[0,\frac{1}{n}\right]\cup\left[\frac{n-1}{n},1\right]} \prod_{j=1}^{n} \sin^2(\pi j x)\,dx = \frac{2}{n}\int_0^1 \prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{n}x\right)dx = 4\int_0^1 h_n(x) e^{2nG(x)}\,dx.$$

Thus equation (3.1) becomes

$$\lim_{n\to\infty} \frac{\int_{\left[0,\frac{1}{n}\right]\cup\left[\frac{n-1}{n},1\right]} \prod_{j=1}^{n} \sin^2(\pi j x)\,dx}{C r^n n^{-1/2}} = 1,$$

which is exactly the statement of Theorem 2.2.

So all that remains is to prove Lemma 3.1. Consider the function

$$u(t) = \log \frac{\sin(\pi t)}{\pi t(1-t)},$$

with $u(0) = u(1) = 0$. This function is continuous and infinitely differentiable on $[0, 1]$, so all its derivatives are also bounded on $[0, 1]$. The Euler–Maclaurin formula yields

$$\sum_{j=0}^{n} u\left(\frac{jx}{n}\right) = \int_0^n u\left(\frac{xt}{n}\right) dt + \frac{u(0)+u(x)}{2} + \frac{x(u'(x)-u'(0))}{12n} - \frac{x^2}{2n^2}\int_0^n B_2(\{t\})u''\left(\frac{xt}{n}\right) dt,$$

where $B_2(y) = y^2 - y + 1/6$ is the second Bernoulli polynomial. Since the integrand in the last integral is bounded (uniformly in x), we get

$$\sum_{j=1}^{n} u\left(\frac{jx}{n}\right) = \int_0^n u\left(\frac{xt}{n}\right) dt + \frac{u(x)}{2} + O\left(\frac{1}{n}\right),$$

using also the fact that $u(0) = 0$. Now we have

$$\begin{aligned}
&\log h_n(x)\\
&= 2\sum_{j=1}^{n} u\left(\frac{jx}{n}\right) - \log(2n) - 2nG(x) + 2\sum_{j=1}^{n} \log\left(\frac{\pi jx}{n}\left(1-\frac{jx}{n}\right)\right)\\
&= 2n\int_0^1 u(xt)\,dt + u(x) - \log(2n) - 2nG(x) + 2\sum_{j=1}^{n} \log\left(\frac{\pi jx}{n}\left(1-\frac{jx}{n}\right)\right) + O\left(\frac{1}{n}\right)\\
&= -2n\int_0^1 \log(\pi xt(1-xt))\,dt + u(x) - \log(2n) + 2\sum_{j=1}^{n} \log\left(\frac{\pi jx}{n}\left(1-\frac{jx}{n}\right)\right) + O\left(\frac{1}{n}\right)\\
&= 4n + \frac{2n}{x}(1-x)\log(1-x) - 2n\log(\pi x) + u(x) - \log(2n) + 2n\log(\pi x/n) + 2\log(n!)\\
&\quad + 2\log\prod_{j=1}^{n}\left(1-\frac{jx}{n}\right) + O\left(\frac{1}{n}\right).
\end{aligned}$$

Now we use Stirling's approximation and simplify to get

$$\log h_n(x) = \log\frac{\sin(\pi x)}{x} + 2n + \left(\frac{2n}{x} - 2n - 1\right)\log(1-x) + 2\log\prod_{j=1}^{n}\left(1-\frac{jx}{n}\right) + O\left(\frac{1}{n}\right). \tag{3.2}$$

Next, note that

$$\prod_{j=1}^{n}\left(1-\frac{jx}{n}\right) = \left(\frac{x}{n}\right)^n \prod_{j=1}^{n}\left(\frac{n}{x} - j\right) = \left(\frac{x}{n}\right)^n \frac{\Gamma(n/x)}{\Gamma(n/x-n)}.$$

Since n/x goes to infinity regardless of the value of x (unlike $n/x - n$, which we will discuss later), we can again apply Stirling's approximation to find

$$\log\prod_{j=1}^{n}\left(1-\frac{jx}{n}\right) = \left(\frac{n}{x} - n\right)\log(n/x) - \frac{n}{x} + \frac{1}{2}\log\left(\frac{2\pi x}{n}\right) - \log\Gamma\left(\frac{n}{x} - n\right) + O\left(\frac{1}{n}\right).$$

Plugging this into (3.2) and simplifying once again yields

$$\log h_n(x) = \log \frac{\sin(\pi x)}{x} + \left(\frac{2n}{x} - 2n - 1\right) \log\left(\frac{n}{x} - n\right) - 2\left(\frac{n}{x} - n\right) + \log(2\pi) - 2\log\Gamma\left(\frac{n}{x} - n\right) + O\left(\frac{1}{n}\right), \tag{3.3}$$

and this estimate is still uniform in x. Now define another function γ by

$$\gamma(t) = t^t e^{-t} \Gamma(t)^{-1} \sqrt{2\pi/t}.$$

This function is continuous on $[0, \infty)$ with $\gamma(0) = 0$ and $\gamma(t) = 1 + O(1/t)$ as $t \to \infty$ by Stirling's formula. Now we can write (3.3) in the more compact form

$$\log h_n(x) = \log \frac{\sin(\pi x)}{x} + 2\log\gamma\left(\frac{n}{x} - n\right) + O\left(\frac{1}{n}\right)$$

or

$$h_n(x) = \frac{\sin(\pi x)}{x} \gamma\left(\frac{n}{x} - n\right)^2 \left(1 + O\left(\frac{1}{n}\right)\right) = \frac{\sin(\pi x)}{x} \gamma\left(\frac{n}{x} - n\right)^2 + O\left(\frac{1}{n}\right),$$

and thus

$$h_n(x) - \frac{\sin(\pi x)}{x} = \frac{\sin(\pi x)}{x} \left(\gamma\left(\frac{n}{x} - n\right)^2 - 1\right) + O\left(\frac{1}{n}\right).$$

All this is still uniform in x. If x is very close to 1, say $x \geqslant n/(n+1)$ (so that $n/x - n \leqslant 1$), then $\sin(\pi x)/x = O(1/n)$, while the second factor on the right-hand side is bounded. If, on the other hand, $x \leqslant n/(n+1)$, then $\gamma(n/x - n)^2 - 1 = O((n/x - n)^{-1})$ and thus

$$\frac{\sin(\pi x)}{x} \left(\gamma\left(\frac{n}{x} - n\right)^2 - 1\right) = O\left(\frac{\sin(\pi x)}{(1-x)n}\right) = O\left(\frac{1}{n}\right).$$

In either case, we have the desired estimate, which completes our proof.

Remark. If we include further terms in the Euler–Maclaurin formula and Stirling's approximation, we find that

$$h_n(x) = \frac{\sin(\pi x)}{x} + \frac{\pi \cos(\pi x)}{6n} + O\left(\frac{1}{n^2}\right),$$

uniformly on compact subsets of $[0, 1)$, in particular on an interval around x_0. This also makes it possible to obtain a more precise asymptotic formula for a_n:

$$a_n = \frac{(4r)^n}{\sqrt{n}} \left(C + \frac{C_1}{n} + O\left(\frac{1}{n^2}\right)\right),$$

where r and C are as in Theorem 2.1 and

$$\begin{aligned} C_1 = {} & \frac{\sqrt{\pi}\sin(\pi x_0)}{12x_0^3(-G''(x_0))^{7/2}}(24G''(x_0)^2 - 12\pi^2 x_0^2 G''(x_0)^2 + 12x_0 G''(x_0)G'''(x_0) \\ & + 5x_0^2 G'''(x_0)^2 - 3x_0^2 G''(x_0)G^{(4)}(x_0)) \\ & + \frac{\pi^{3/2}\cos(\pi x_0)(6G''(x_0) + 2x_0^2 G''(x_0)^2 + 3x_0 G'''(x_0))}{3x_0^2(-G''(x_0))^{5/2}} \\ = {} & 0.0262451044\ldots. \end{aligned}$$

In principle, one can extend the asymptotic formula further to include arbitrarily many terms.

4. Error bounding: Proof of Theorem 2.3

We now begin the proof of Theorem 2.3, which says that

$$\int_{1/n}^{(n-1)/n} \prod_{j=1}^{n} \sin^2(\pi j x)\, dx = O(\rho^n) \quad \text{for some } \rho < r.$$

The following result will imply Theorem 2.3 immediately.

Theorem 4.1. *Let r be defined as in Theorem 2.1. Then there exist $C < \log r$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have*

$$\frac{1}{n}\sum_{j=1}^{n} \log(\sin^2(\pi j x)) \leqslant C \quad \textit{for all } x \in [1/n, (n-1)/n].$$

To prove Theorem 4.1, we will approximate each $x \in [1/n, (n-1)/n]$ with a rational. Dirichlet's theorem provides exactly the sharp approximation that we require.

Theorem 4.2 (Dirichlet: see [16]). *Let $x \in \mathbb{R}$, $n \in \mathbb{N}$. Then there exist coprime $p_{n,x}, q_{n,x} \in \mathbb{Z}$, where $1 \leqslant q_{n,x} \leqslant n$, such that*

$$\left| x - \frac{p_{n,x}}{q_{n,x}} \right| \leqslant \frac{1}{q_{n,x}(n+1)}. \tag{4.1}$$

To begin, we fix $n \in \mathbb{N}$ and $x \in [1/n, (n-1)/n]$. Let coprime $p_{n,x}, q_{n,x} \in \mathbb{Z}$ be as described in Theorem 4.2. Without loss of generality,

$$x \geqslant \frac{p_{n,x}}{q_{n,x}},$$

because otherwise we can (instead) use the values

$$1 - x \quad \text{and} \quad \frac{q_{n,x} - p_{n,x}}{q_{n,x}},$$

since $\sin^2(\pi jt)$ is the same for both x and $1-x$, and since the bound in Theorem 4.2 will still hold. We then set

$$y := x - \frac{p_{n,x}}{q_{n,x}},$$

so that we have

$$x = \frac{p_{n,x}}{q_{n,x}} + y, \quad \text{where} \quad 0 \leqslant y \leqslant \frac{1}{q_{n,x}(n+1)}. \tag{4.2}$$

Henceforth we suppress the (n,x) subscript, and simply write $p := p_{n,x}$ and $q := q_{n,x}$ (but we must keep in mind that these values actually depend on n and x).

Now, our overall goal is to bound the quantity

$$L := \frac{1}{n}\sum_{j=1}^{n} \log(\sin^2(\pi jx)),$$

and we are approximating x with the rational p/q. So it makes sense to partition the values of $j = 1 \ldots n$ by their equivalence classes modulo q. If we define $k := \lceil n/q \rceil - 1$ and $s := n \bmod q$, then we have

$$\begin{aligned} L &= \frac{1}{n}\sum_{m=0}^{k-1}\sum_{\ell=1}^{q} \log\left(\sin^2\left(\pi(mq+\ell)\left(\frac{p}{q}+y\right)\right)\right) + \frac{1}{n}\sum_{\ell=1}^{s} \log\left(\sin^2\left(\pi(kq+\ell)\left(\frac{p}{q}+y\right)\right)\right) \\ &= \frac{1}{n}\sum_{m=0}^{k-1}\sum_{\ell=1}^{q} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(mq+\ell)y\right)\right)\right) + \frac{1}{n}\sum_{\ell=1}^{s} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(kq+\ell)y\right)\right)\right). \end{aligned} \tag{4.3}$$

This is the basic paradigm under which we will operate for the remainder of our proof. We now break the argument into two cases, depending on the relative sizes of q and n.

Case 1: $q \leqslant \sqrt{n}$. The basic idea here will be to form Riemann sums at each $\ell p/q$. Dropping the remainder term in (4.3) (which we can do since $\log(\sin^2(\cdot))$ is always non-positive), yields

$$L \leqslant \frac{1}{n}\sum_{\ell=1}^{q}\sum_{m=0}^{k-1} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(mq+\ell)y\right)\right)\right) \tag{4.4}$$

$$= \frac{1}{nqy}\sum_{\ell=1}^{q}\left[qy\sum_{m=0}^{k-1} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+\ell y+mqy\right)\right)\right)\right]. \tag{4.5}$$

We note that the inner bracketed terms (4.5) look like Riemann sums. And if we remove, for fixed ℓ, half the 0th term and half the $(k-1)$st, what remains will be a trapezoidal sum for

$$\int_{\ell y}^{\ell y+(k-1)qy} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+t\right)\right)\right) dt.$$

The trapezoidal sum does not exceed the value of the integral, since the integrand is concave. And since the integrand is non-positive, we can shorten the region of integrating

without increasing the value. Thus,

$$L \leqslant \frac{1}{nqy} \sum_{\ell=1}^{q} \int_{\ell y}^{\ell y+(k-1)qy} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+t\right)\right)\right) dt \tag{4.6}$$

$$\leqslant \frac{1}{nqy} \sum_{\ell=1}^{q} \int_{qy}^{(k-1)qy} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+t\right)\right)\right) dt. \tag{4.7}$$

Now, since p and q are relatively prime, the quantity $\ell p/q \bmod 1$ will range over all values

$$\left\{\frac{1}{q}, \frac{2}{q}, \ldots, 1\right\}$$

as ℓ ranges from 1 to q. Then, by the identity

$$\prod_{j=1}^{q} \sin\left(\pi\left(\frac{j}{q}+t\right)\right) = -\frac{\sin(\pi q t)}{2^{q-1}}, \tag{4.8}$$

inequality (4.7) implies

$$\begin{aligned} L &\leqslant \frac{1}{nqy} \int_{qy}^{(k-1)qy} \log\left(\frac{\sin^2(\pi q t)}{4^{q-1}}\right) dt \\ &= \frac{(k-2)(q-1)}{n} \log\left(\frac{1}{4}\right) + \frac{1}{nq^2y} \int_{q^2y}^{(k-1)q^2y} \log(\sin^2(\pi t))\, dt \\ &= \frac{k-2}{n}\left[(q-1)\log\left(\frac{1}{4}\right) + \frac{1}{(k-2)q^2y} \int_{q^2y}^{(k-1)q^2y} \log(\sin^2(\pi t))\, dt\right]. \end{aligned} \tag{4.9}$$

And since $k = \lceil n/q \rceil - 1$ and $q \leqslant \sqrt{n}$, we have

$$\frac{k-2}{n} = \frac{1}{q} + O(n^{-1}) = \frac{1}{q}(1 + O(n^{-1/2})).$$

Plugging that into (4.9), we obtain

$$L \leqslant \left(1 + O(n^{-1/2})\right)\left[\frac{q-1}{q}\log\left(\frac{1}{4}\right) + \frac{1}{q}\frac{1}{(k-2)q^2y} \int_{q^2y}^{(k-1)q^2y} \log(\sin^2(\pi t))\, dt\right]. \tag{4.10}$$

From here our strategy will be to transform the integral on the right-hand side of (4.10) into a value of the function $2G(x) = \int_0^1 \log(\sin^2(\pi x t))\, dt$. We note that

$$\frac{1}{(k-2)q^2y} \int_{q^2y}^{(k-1)q^2y} \log(\sin^2(\pi t))\, dt$$

is an average-value integral, whose integrand $\log(\sin^2(\pi t))$ is concave with a unique maximum at $t = 1/2$. So if $(k-1)q^2y < 1/8$ (say), then since $1/8 < 1/2$ and $\log(\sin^2(\pi/8)) < \log(1/4)$, we have

$$L \leqslant \left(1 + O(n^{-1/2})\right)\left[\frac{q-1}{q}\log\left(\frac{1}{4}\right) + \frac{1}{q}\log\left(\sin^2\left(\frac{\pi}{8}\right)\right)\right] \leqslant \left(1 + O(n^{-1/2})\right)\log\left(\frac{1}{4}\right).$$

Since $1/4 < r = 0.3719264606\ldots$, and thus

$$\log\left(\frac{1}{4}\right) = -1.386294361\cdots < \log r = -0.9890591305\ldots,$$

we are in this case done. So we may assume that $(k-1)q^2y \geqslant 1/8$. In this case we can extend the lower limit of the integral in (4.10) to $t = 0$ for the price of an $O(n^{-1/4})$ term. For we have

$$\begin{aligned}\frac{1}{(k-2)q^2y}\int_0^{q^2y}\log(\sin^2(\pi t))\,dt &= \left[\frac{1}{(k-2)\sqrt{q^2y}}\right]\frac{1}{\sqrt{q^2y}}\int_0^{q^2y}\log(\sin^2(\pi t))\,dt\\ &= \frac{1}{(k-2)\sqrt{q^2y}}O(1)\\ &= O\left(\frac{\sqrt{8(k-1)}}{(k-2)}\right)\\ &= O(n^{-1/4}),\end{aligned} \tag{4.11}$$

where the $O(1)$ bound in (4.11) is easily verified through L'Hôpital's rule, and the $O(n^{-1/4})$ bound follows from the fact that $k = \lfloor n/q \rfloor$ where $q \leqslant \sqrt{n}$.

Then, operating on the integral in (4.10), we have

$$\begin{aligned}\frac{1}{(k-2)q^2y}\int_{q^2y}^{(k-1)q^2y}\log(\sin^2(\pi t))\,dt &= \frac{1}{(k-2)q^2y}\int_0^{(k-1)q^2y}\log(\sin^2(\pi t))\,dt + O(n^{-1/4})\\ &\leqslant \frac{1}{(k-1)q^2y}\int_0^{(k-1)q^2y}\log(\sin^2(\pi t))\,dt + O(n^{-1/4})\\ &= \int_0^1\log(\sin^2(\pi(k-1)q^2yt))\,dt + O(n^{-1/4})\\ &= 2G((k-1)q^2y) + O(n^{-1/4}) \leqslant \log r + O(n^{-1/4}).\end{aligned}$$

Plugging into (4.10), we obtain

$$\begin{aligned}L &\leqslant \left(1 + O(n^{-1/2})\right)\left[\frac{q-1}{q}\log\left(\frac{1}{4}\right) + \frac{1}{q}(\log r + O(n^{-1/4}))\right]\\ &\leqslant \left(1 + O(n^{-1/4})\right)\left[\frac{\log(1/4) + \log r}{2}\right].\end{aligned}$$

We note here that q cannot be 1, since for $x \in [1/n, (n-1)/n]$, the distance to the nearest integer is at least $1/n > 1/(n+1)$, contradicting (4.1). Thus $q \geqslant 2$. Since $\log(1/4) < \log r$, this completes our consideration of the case when $q \leqslant \sqrt{n}$.

Case 2: $q > \sqrt{n}$. In Case 1 we took Riemann sums at each i/q. In this case we will let the values $0, 1/q, \ldots (q-1)/q$ form a Riemann sum that ranges over all of $[0, 1]$. We begin by

referring back to the canonical representation of (4.3),

$$L = \frac{1}{n}\sum_{m=0}^{k-1}\sum_{\ell=1}^{q}\log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(mq+\ell)y\right)\right)\right) + \frac{1}{n}\sum_{\ell=1}^{s}\log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(kq+\ell)y\right)\right)\right), \tag{4.12}$$

where $s = n \bmod q$, and in Case 1 we discarded the right-hand remainder sum at the very beginning. We will not be able to do so in this case.

We summarize our approach to Case 2 in the following proposition.

Proposition 4.3. *For fixed $m \leqslant k$ and fixed i, $1 \leqslant i \leqslant q$, there is at most one ℓ, $1 \leqslant \ell \leqslant q$, such that*

$$\left(\frac{\ell p}{q}+(mq+\ell)y\right) \bmod 1 \ \in \left[\frac{i-1}{q}, \frac{i}{q}\right).$$

Furthermore, if $m < k$, there is exactly one such ℓ.

Given $m < k$ and ℓ such that

$$\left(\frac{\ell p}{q}+(mq+\ell)y\right) \bmod 1 \in \left[\frac{i-1}{q}, \frac{i}{q}\right),$$

we define

$$z_{m,i} := \left(\frac{\ell p}{q}+(mq+\ell)y\right) \bmod 1.$$

The content of this proposition follows immediately from the Dirichlet bound of (4.2) and the relative primality of p and q.

Now, for fixed m, the points $z_{m,1}, \ldots, z_{m,q}$ will *not* be equidistant. However, by the concavity and symmetry of the function $\log(\sin^2(\pi(\cdot)))$, we will always have either

$$\log(\sin^2(\pi z_{m,i})) \leqslant \log\left(\sin^2\left(\frac{\pi i}{q}\right)\right)$$

or

$$\log(\sin^2(\pi z_{m,i})) \leqslant \log\left(\sin^2\left(\frac{\pi(i-1)}{q}\right)\right),$$

depending on whether $i/q < 1/2$ or $i/q > 1/2$. (We will not have such a bound in the case where $(i-1)/q < 1/2 < i/2$, but the midpoint takes care of itself, since $\log(\sin^2(\pi/2)) = 0$.) So we have

$$\frac{1}{n}\sum_{m=0}^{k-1}\sum_{\ell=1}^{q}\log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(mq+\ell)y\right)\right)\right) \leqslant \frac{1}{n}\sum_{m=0}^{k-1}\sum_{i=1}^{q-1}\log\left(\sin^2\left(\frac{\pi i}{q}\right)\right) = \frac{k}{n}\log\left(\frac{q^2}{4^{q-1}}\right),$$

where the final equality follows from a variant on the sine-identity (4.8). Plugging this inequality into our base-expression (4.12), we obtain

$$L \leqslant \frac{k}{n}\log\left(\frac{q^2}{4^{q-1}}\right) + \frac{1}{n}\sum_{\ell=1}^{s}\log\left(\sin^2\left(\pi\left(\frac{\ell p}{q} + (kq+\ell)y\right)\right)\right). \tag{4.13}$$

We note that we still have a strange remainder sum lurking at the end of (4.13). We will handle this remainder sum in two separate cases.

Case 2a: $s/q \in [1/2, 3/5]$. This is the hard case; Case 2b will follow very easily from it. We begin by noting that by Proposition 4.3, each of the values $(\ell p/q + (kq+\ell)y) \bmod 1$ will lie in a different interval $[(i-1)/q, i/q)$. The difficulty here is that since ℓ will only range from 1 to s, not all of these intervals will contain such a point. To handle this problem, we rename and order the set $\{(\ell p/q + (kq+\ell)y) \bmod 1 : \ell = 1 \ldots s\}$ as $\{\beta_1, \beta_2, \ldots, \beta_s\}$, where the points are ordered by increasing distance from the point $x = 1/2$. To avoid considering separate cases for odd and even q, we ignore the first point β_1. But we must have

$$\log(\sin^2(\pi\beta_{2j})) \leqslant \log\left(\sin^2\left(\pi\left(\frac{1}{2} - \frac{j-1}{q}\right)\right)\right), \quad \text{and identically for } \beta_{2j+1}, \tag{4.14}$$

for $j = 1 \ldots \lfloor s/2 \rfloor$. The underlying logic here is that the second and third points can be no closer to $1/2$ than 0, the fourth and fifth can be no closer than $1/q$, and so on, since they all lie in separate intervals $[(i-1)/q, i/q]$ and are ordered by their distances from $1/2$.

So we have the global approximation

$$\sum_{j=1}^{s}\log(\sin^2(\pi\beta_\ell)) \leqslant 2\sum_{j=1}^{\lfloor s/2 \rfloor}\log\left(\sin^2\left(\pi\left(\frac{1}{2} - \frac{j-1}{q}\right)\right)\right).$$

However, it will be necessary to consider how much error we accumulate in carrying it out. The following lemma will give us a means of bounding this error from below.

Lemma 4.4. *Under the hypotheses of Case 2a, there is some positive integer N such that whenever $n > N$, at least $q/12$ of the points $\{(\ell p/q + \ell y + kqy) \bmod 1\}_{\ell=1}^{s}$ lie in the region $A = [0, 1/10] \cup [9/10, 1]$.*

Proof. Asymptotically, $q/5$ of the *intervals*

$$\left\{\left[\frac{\ell p \bmod q}{q}, \frac{(\ell p + 1) \bmod q}{q}\right]\right\}_{\ell=1}^{q}$$

will lie strictly within A. So we can find N such that at least $q/6$ of these intervals lie in A whenever $n > N$. Now we know that for every i, $1 \leqslant i \leqslant q$ such that $[(i-1)/q, i/q] \subset A$, there is a unique $\ell_i, 1 \leqslant \ell_i \leqslant q$ such that $\ell_i p \bmod q = i - 1$. If it happens that this $\ell_i > s$, then since $s \geqslant q/2$ by the hypotheses of Case 2a, we must have $q - \ell_i \leqslant s$. Then $(q - \ell_i)p \bmod q = q - i + 1$, and since $i/q \in A$ we must have $q - i/q \in A$ also, by the symmetry of A. It is possible that $(q - i + 1)/q$ will not be in A despite $(q - i)/q$'s being in there, but this can happen for at most one i; this contingency is therefore asymptotically

negligible. So asymptotically at least half the $\ell \leqslant q$ such that

$$\left[\frac{\ell p \bmod q}{q}, \frac{(\ell p+1) \bmod q}{q}\right] \subset A$$

will also satisfy $\ell \leqslant s$; so there will be at least $(1/2)(q/6) = q/12$ such ℓ.

This lemma says that when we make the $s-1$ substitutions described in (4.14), at least $q/12$ of the β_j we substitute for will lie in A. And since in each case our substituted value will be $1/2-(j-1)/q$ for some $1 \leqslant j \leqslant \lfloor s/2 \rfloor$, for at least $q/12$ points, we will have an error of at least

$$\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right).$$

We then have

$$\begin{aligned}
&\frac{1}{n}\sum_{\ell=1}^{s} \log\left(\sin^2\left(\pi\left(\frac{\ell p}{q}+(kq+\ell)y\right)\right)\right) \qquad (4.15)\\
&\quad\leqslant \frac{2}{n}\sum_{\ell=1}^{\lfloor s/2\rfloor} \log\left(\sin^2\left(\pi\left(\frac{1}{2}-\frac{\ell-1}{q}\right)\right)\right)+\frac{q}{12n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)\\
&\quad\leqslant \frac{2q}{n}\int_{\frac{1}{2}-\frac{s-1}{2q}+\frac{1}{q}}^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt+\frac{q}{12n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)\\
&\quad= \frac{2q}{n}\int_{\frac{1}{2}-\frac{s}{2q}}^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt+\frac{q}{12n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)+O\left(\frac{\log n}{n}\right), \qquad (4.16)
\end{aligned}$$

where we get the second inequality by removing half the first and last terms to make a trapezoid sum. Plugging (4.15) into (4.13), we obtain

$$\begin{aligned}
L &\leqslant \frac{k}{n}\log\left(\frac{q^2}{4^{q-1}}\right)+\frac{2q}{n}\int_{\frac{1}{2}-\frac{s}{2q}}^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt+\frac{q}{12n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)+O\left(\frac{\log n}{n}\right)\\
&= \frac{2k(q-1)}{n}\log\left(\frac{1}{2}\right)+\frac{2q}{n}\int_{\frac{1}{2}-\frac{s}{2q}}^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt+\frac{q}{12n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)+O\left(\frac{\log n}{\sqrt{n}}\right). \qquad (4.17)
\end{aligned}$$

Now, since

$$\log\left(\frac{1}{2}\right)=\int_0^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt,$$

by symmetry of $\log(\sin^2(\pi t))$ about $t = 1/2$ we can write

$$\begin{aligned}\frac{2q}{n}\log\left(\frac{1}{2}\right) + \frac{2q}{n}\int_{\frac{1}{2}-\frac{s}{2q}}^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt &= \frac{2q}{n}\int_0^{\frac{1}{2}+\frac{s}{2q}} \log(\sin^2(\pi t))\,dt \\ &= \frac{2q}{n}\frac{\frac{1}{2}+\frac{s}{2q}}{\frac{1}{2}+\frac{s}{2q}}\int_0^{\frac{1}{2}+\frac{s}{2q}} \log(\sin^2(\pi t))\,dt \\ &= \frac{q+s}{n}\cdot 2G\left(\frac{1}{2}+\frac{s}{2q}\right) \leqslant \frac{q+s}{n}\log r.\end{aligned}$$

Plugging into (4.17), we obtain

$$\begin{aligned}L &\leqslant \frac{2k(q-1)}{n}\log\left(\frac{1}{2}\right) - \frac{2q}{n}\log\left(\frac{1}{2}\right) + \frac{q+s}{n}\log r + \frac{q}{12n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right) + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &\leqslant \frac{2(k-1)q}{n}\log\left(\frac{1}{2}\right) + \frac{q+s}{n}\log r + \frac{q+s}{24n}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right) + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &= \frac{(k-1)q}{n}\log\left(\frac{1}{4}\right) + \frac{q+s}{n}\left(\log r + \frac{1}{24}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)\right) + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &\leqslant \frac{(k-1)q+q+s}{n}\max\left(\log\left(\frac{1}{4}\right), \log r + \frac{1}{24}\log\left(\frac{\sin^2(\frac{\pi}{10})}{\sin^2(\frac{\pi}{5})}\right)\right) + O\left(\frac{\log n}{\sqrt{n}}\right).\end{aligned}$$

Since $(k-1)q + q + s = kq + s = n$ and both constants inside the maximum are strictly less than $\log r$, this concludes our consideration of Case 2a.

Case 2b: $s/q \notin [1/2, 3/5]$. In this case we follow the same procedure as in Case 2a, but do not have to trouble ourselves about the error accumulated when we shift our β_i. We pick up at (4.17) with the $q/(12n)$ bit removed:

$$\begin{aligned}L &\leqslant \frac{2k(q-1)}{n}\log\left(\frac{1}{2}\right) + \frac{2q}{n}\int_{\frac{1}{2}-\frac{s}{2q}}^{\frac{1}{2}} \log(\sin^2(\pi t))\,dt + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &= \frac{(k-1)q}{n}\log\left(\frac{1}{4}\right) + \frac{q+s}{n}\cdot 2G\left(\frac{1}{2}+\frac{s}{2q}\right) + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &\leqslant \frac{(k-1)q}{n}\log\left(\frac{1}{4}\right) + \frac{q+s}{n}\cdot\left[\sup_{t\notin\left[\frac{3}{4},\frac{4}{5}\right]} 2G(t)\right] + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &\leqslant \frac{(k-1)q+q+s}{n}\max\left(\log\left(\frac{1}{4}\right), \sup_{t\notin\left[\frac{3}{4},\frac{4}{5}\right]} 2G(t)\right) + O\left(\frac{\log n}{\sqrt{n}}\right).\end{aligned}$$

Again, we are done since both $\log(1/4)$ and the sup are fixed constants $< \log r$, as in the previous case. This concludes our consideration of Case 2b, proving Theorem 4.1. Theorem 2.3 immediately follows, and we are done. □

5. Final remarks

Shaun Stevens recently asked [17] how fast or slow the asymptotic growth of

$$\int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi b_j x)\,dx$$

can be, for general b_j (we have analysed the $b_j = j$ case). He conjectured that the fastest growth is for $b_j = 1$ (which, as he notes, gives growth rate $2^{2n}/\sqrt{\pi n}$, using Stirling's approximation). This is indeed true: to prove it, we require a version of the rearrangement inequality for integrals. For a measurable function $f : \mathbb{R} \to [0,\infty)$, the *symmetric decreasing rearrangement* f^* is the symmetric decreasing function whose level sets have the exact same size: for $A \subseteq \mathbb{R}$, let A^* be the interval $(-\lambda(A)/2, \lambda(A)/2)$, where λ denotes the Lebesgue measure. Moreover, set $L_f(t) = \{y \,:\, f(y) > t\}$, and let $I_{L_f^*(t)}$ be the indicator function of $L_f^*(t)$. Then f^* is given by

$$f^*(x) = \int_0^\infty I_{L_f^*(t)}(x)\,dt = \sup\{t \,:\, x \in L_f^*(t)\}.$$

Note that $L_{f^*}(t) = L_f^*(t)$ and thus $\lambda(L_{f^*}(t)) = \lambda(L_f(t))$ for all $t \geqslant 0$. The following theorem is a special case of [8, Theorem 3.8]. If $f_1, f_2, \ldots$ are L^∞-functions from $\mathbb{R}$ to $[0,\infty)$ with compact support, then

$$\int_{-\infty}^{\infty} f_1(x)f_2(x)\cdots f_n(x)\,dx \leqslant \int_{-\infty}^{\infty} f_1^*(x)f_2^*(x)\cdots f_n^*(x)\,dx.$$

It is not difficult to see that the symmetric decreasing rearrangement of $f(x) = \sin(\pi b x)^2$ on $[0,1]$ is $f^*(x) = \cos(\pi x)^2$ with support $(-1/2, 1/2)$ for every positive integer b. So we obtain

$$\int_0^1 \prod_{j=1}^{n} 4\sin^2(\pi b_j x)\,dx \leqslant 4^n \int_{-1/2}^{1/2} \cos^{2n}(\pi x)\,dx = \binom{2n}{n},$$

with equality whenever the sequence $b_1, b_2, \ldots, b_n$ is constant.

We are also eager to better understand the asymptotic growth of

$$a_{n,k} = [x^k]\prod_{j=1}^{n}(1-x^j)(1-x^{-j}) = \int_0^1 e^{-2\pi i k x}\prod_{j=1}^{n} 4\sin^2(\pi j x)\,dx.$$

As discussed in the Introduction, the a_ns throughout this paper correspond to the case $k = 0$. If $k/n \to a$ as $n \to \infty$, then the proof of Theorem 2.1 can be modified easily to prove that

$$\frac{a_{n,k}}{a_n} = \frac{a_{n,k}}{a_{n,0}} \to \cos(2\pi a x_0).$$

Things get more complicated when k is even larger, though, so that one probably needs to distinguish different cases.

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